A REMARK ON THE HIGHER CAPELLI IDENTITIES

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Abstract
A simple proof of the higher Capelli identities is given.

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The aim of this note is to give a simple proof of the remarkable generalizations of the classical Capelli identity discovered by A. Okounkov [O1] and also proved in different ways by him [O2] and M. Nazarov [N]. Following [O2], we use some properties of the Jucys–Murphy elements in the group algebra for the symmetric group, together with the branching properties of the Young basis. The difference between our proof and that of [O2] is that in our approach we do not need to use the Wick formula and the Olshanskiĭ special symmetrization map.

Denote by $\mathcal{PD}$ the algebra of polynomial coefficient differential operators in mn variables x_{ai}, where $a = 1, \dots, m$ and $i = 1, \dots, n$. Consider the representation of the Lie algebra $\mathfrak{gl}(m)$ by differential operators defined on the standard generators E_{ab} as follows:

$$E_{ab} = \sum_{i=1}^{n} x_{ai} \partial_{bi}, \tag{1}$$

where $\partial_{ai} := \partial/\partial x_{ai}$. Let E, X, D denote the formal matrices with the entries E_{ab}, x_{ai}, ∂_{ai}, respectively. Then (1) can be written in a matrix form as follows:

$$E = XD', \tag{2}$$

where D' is the matrix transposed to D.

As in [O2], for a partition λ and a standard λ-tableau T we denote by v_T the corresponding vector of the Young orthonormal basis in the irreducible representation V^λ of the symmetric group S_k, $k = |\lambda|$. We let $c_T(r) = j - i$ if the cell $(i, j) \in \lambda$ is occupied by the entry r of the tableau T. Introduce the matrix element

$$\Psi_{TT'} = \sum_{s \in S_k} (s \cdot v_T, v_{T'}) \cdot s^{-1} \in \mathbb{C}[S_k]. \tag{3}$$

The symmetric group S_k acts in a natural way in the tensor space $(\mathbb{C}^m)^{\otimes k}$, so that we can identify permutations from S_k with elements of the algebra

$$\underbrace{\mathrm{Mat}_{mm} \otimes \cdots \otimes \mathrm{Mat}_{mm}}_{k}, \tag{4}$$

where by Mat_{pq} we denote the space of $p \times q$-matrices; for $p = q$ we also regard it as an algebra. By P_{ij} we denote the element of (4) corresponding to the transposition $(ij) \in S_k$.

We regard the tensor product $A \otimes B \otimes \cdots \otimes C$ of k matrices A, B, …, C of size $p \times q$ with entries from an algebra $\mathcal{A}$ as an element

$$\sum A_{a_1 i_1} B_{a_2 i_2} \cdots C_{a_k i_k} \otimes e_{a_1 i_1} \otimes e_{a_2 i_2} \otimes \cdots \otimes e_{a_k i_k} \in \mathcal{A} \otimes (\mathrm{Mat}_{pq})^{\otimes k},$$

where the e_{ai} are the standard matrix units.

Theorem [O2, N]**.** *Let T and T' be two standard tableaux of the same shape. Then*

$$(E - c_T(1)) \otimes \cdots \otimes (E - c_T(k)) \cdot \Psi_{TT'} = X^{\otimes k} \cdot (D')^{\otimes k} \cdot \Psi_{TT'} \tag{5}$$

Proof. We use induction on k. Denote by U the tableau obtained from T by removing the cell with the entry k. Using the branching property of the Young basis $\{v_T\}$ one can easily check that

$$\Psi_{TT'} = \mathrm{const} \cdot \Psi_{UU}\Psi_{TT'},$$

where 'const' is a nonzero constant (more precisely, const $= \dim\mu/(k-1)!$ where μ is the shape of U and $\dim\mu = \dim V^{\mu}$).

So, we can rewrite the left hand side of (5) as follows:

$$\mathrm{const} \cdot (E - c_T(1)) \otimes \cdots \otimes (E - c_T(k-1)) \cdot \Psi_{UU} \otimes (E - c_T(k)) \cdot \Psi_{TT'}.$$

By the induction hypothesis, this equals

$$\mathrm{const} \cdot X^{\otimes k-1} \cdot (D')^{\otimes k-1} \cdot \Psi_{UU} \otimes (XD' - c_T(k)) \cdot \Psi_{TT'}$$

$$= X^{\otimes k-1} \cdot (D')^{\otimes k-1} \otimes (XD' - c_T(k)) \cdot \Psi_{TT'}$$

$$= \Big(\sum x_{a_1 i_1} \cdots x_{a_{k-1} i_{k-1}} \partial_{b_1 i_1} \cdots \partial_{b_{k-1} i_{k-1}} (x_{a_k i_k} \partial_{b_k i_k} - \frac{\delta_{a_k b_k}}{n} c_T(k))$$

$$\otimes e_{a_1 b_1} \otimes \cdots \otimes e_{a_k b_k}\Big) \cdot \Psi_{TT'}.$$

Now we transform this expression using the relations $\partial_{bj} x_{ai} = x_{ai}\partial_{bj} + \delta_{ab}\delta_{ij}$ to obtain

$$\Big(\sum x_{a_1 i_1} \cdots x_{a_k i_k} \partial_{b_1 i_1} \cdots \partial_{b_k i_k} \otimes e_{a_1 b_1} \otimes \cdots \otimes e_{a_k b_k}\Big) \cdot \Psi_{TT'}$$
$$+ \Big(\sum x_{a_1 i_1} \cdots x_{a_{k-1} i_{k-1}} \partial_{b_1 i_1} \cdots \partial_{b_{k-1} i_{k-1}} \otimes e_{a_1 b_1} \otimes \cdots \otimes e_{a_{k-1} b_{k-1}} \otimes 1\Big)$$
$$\times (P_{1k} + \cdots + P_{k-1,k} - c_T(k)) \cdot \Psi_{TT'}. \qquad (6)$$

Note that $P_{1k} + \cdots + P_{k-1,k}$ is the image of the Jucys–Murphy element $(1k) + \cdots + (k-1,k)$. It has the property

$$((1k) + \cdots + (k-1,k)) \cdot \Psi_{TT'} = c_T(k) \cdot \Psi_{TT'},$$

which was also used in [O2] and can be easily derived from the following formula due to Jucys and Murphy:

$$((1k) + \cdots + (k-1,k)) \cdot v_T = c_T(k) \cdot v_T.$$

This proves that the second summand in (6) is zero, which completes the proof of Theorem.

It was shown in [O2] that taking trace in both sides of (5) over all tensor factors Mat_{mm}, one obtains the following 'higher Capelli identities' proved in [O1] and [N].

Corollary. *For any standard tableau T of shape λ one has*

$$\mathrm{tr}\,(E - c_T(1)) \otimes \cdots \otimes (E - c_T(k)) \cdot \Psi_{TT} = \frac{1}{\dim \lambda}\,\mathrm{tr}\, X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\lambda, \qquad (7)$$

where

$$\chi^\lambda = \sum_{s \in S_k} \chi^\lambda(s) \cdot s \in \mathbb{C}[S_k]$$

is the character of V^λ.

In particular, in the case $\lambda = (1^k)$ one obtains the classical Capelli identity.

The left hand side of (7) is an element of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m))$. Due to (7), it depends only on the partition λ and does not depend on the tableau T. Moreover, the set of these elements, where λ runs over the partitions with length $\leq m$ forms a basis in the center. Their eigenvalues in highest weight representations of $\mathfrak{gl}(m)$ are the 'shifted Schur polynomials'; see [O1, O2, OO, N] for details.

References

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